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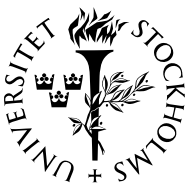

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Statistics of the Information Component in Bayesian Neural Networks

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Abstract

In previous work approximate solutions have been used for expectation and variance of the *information component (IC)*. This report presents an analytical approach to calculate exact expressions for the expectation and variance of the information component (IC). The IC is used in a Bayesian neural network [3] as a weight between neurons representing discrete events. The IC relates the information possessed about one state of one variable with one state of another variable, and is used for calculation of a posterior probability distribution conditioned on a set of given input events. It is used as a measure of disproportionality in data mining [1]. The mutual information between two variables, as defined in information theory [4], can in its discrete form be regarded as a weighted sum of ICs. The expectation of the IC provides a measure of the strength of an association between two states and its variance a measure of the uncertainty, which is essential for low counter values.

1 Introduction

We are concerned with computing the expectation and variance of

$$IC_{ij} = \log \frac{\mathbf{p}_{ij}}{\mathbf{p}_i \cdot \mathbf{p}_j}.$$

We consider $\mathbf{p}_{ij}$, $\mathbf{p}_i$ and $\mathbf{p}_j$ as random variables. We know or assume that the marginal distributions of these variables are $Beta$-distributions with respective parameters.

2 Expectation of $\log \mathbf{p}$

To compute the expectation of IC_{ij} we observe that

$$E\left[IC_{ij}\right] = E \log \mathbf{p}_{ij} - E \log \mathbf{p}_i - E \log \mathbf{p}_j,$$

which means that the marginal distributions are sufficient to compute the required expectations. We do this in a generic manner, i.e. we consider $E \log \mathbf{p}$ when $\mathbf{p} \in B(a,b)$, where $B(a,b)$ is the $Beta$ -distribution with parameters a and b. This means that $\mathbf{p}$ has the probability density ([6, p. 173])

$$f_{\mathbf{p}}(p) = \frac{\Gamma(a+b)}{\Gamma(a) \cdot \Gamma(b)} p^{a-1} \cdot (1-p)^{b-1}, \tag{2.1}$$

where $\Gamma(z)$ is Euler's gamma function. The $Beta$ -function is

$$B(a,b) = \frac{\Gamma(a) \cdot \Gamma(b)}{\Gamma(a+b)} \tag{2.2}$$

so that

$$f_{\mathbf{p}}(p) = \frac{1}{B(a,b)} p^{a-1} \cdot (1-p)^{b-1}. \tag{2.3}$$

By a standard result about expectation of functions of random variables we have

$$E \log \mathbf{p} = \frac{1}{B(a,b)} \int_0^1 \log p \cdot p^{a-1} \cdot (1-p)^{b-1} dp. \tag{2.4}$$

Next we prove the following

Proposition 2.1 $$E \log \mathbf{p} = \frac{\Gamma'(a)}{\Gamma(a)} - \frac{\Gamma'(a+b)}{\Gamma(a+b)}.$$

Proof: In view of (2.4) we compute first the integral $\int_0^1 \log p \cdot p^{a-1} \cdot (1-p)^{b-1} dp$. This is done as follows. We recall that if $\frac{d}{da}$ denotes differentiation with respect to a then

$$\frac{d}{da} x^a = \frac{d}{da} e^{a \log x} = \log x \cdot x^a, \tag{2.6}$$

using the natural logarithms. Thus, since

$$\int_0^1 p^{a-1} \cdot (1-p)^{b-1} dp = B(a,b),$$

we have from (2.6) that

$$\int_0^1 \log p \cdot p^{a-1} \cdot (1-p)^{b-1} dp = \int_0^1 \frac{d}{da} p^a \cdot p^{-1} \cdot (1-p)^{b-1} dp = \frac{\partial}{\partial a} B(a,b). \tag{2.7}$$

assuming it is justified to exchange the order of the differentiation and the integral. Next we observe that if $f'(x)$ is the first derivative of $f(x)$, then

$$\frac{d}{dx} \log f(x) = \frac{f'(x)}{f(x)} \tag{2.8}$$

and

$$\begin{aligned} E \log \mathbf{p} &= \frac{1}{B(a,b)} \int_0^1 \log p \cdot p^{a-1} \cdot (1-p)^{b-1} dp \\ &= \frac{1}{B(a,b)} \cdot \frac{\partial}{\partial a} B(a,b) \qquad (2.9) \\ &= \frac{\partial}{\partial a} \log B(a,b). \qquad (2.10) \end{aligned}$$

By (2.2) we have

$$\frac{\partial}{\partial a} \log B(a,b) = \frac{\partial}{\partial a} \log \frac{\Gamma(a) \cdot \Gamma(b)}{\Gamma(a+b)}. \tag{2.11}$$

By straightforward differentiation this becomes

$$\frac{\partial}{\partial a} \log B(a,b) = \frac{\Gamma'(a)}{\Gamma(a)} - \frac{\Gamma'(a+b)}{\Gamma(a+b)}. \tag{2.12}$$

In other words we have obtained (2.5) above as claimed. □

2.1 A series formula

By a well known formula recapitulated and proved in [5, p. 467] it holds for z which is not a negative integer that

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{z+n} \right), \tag{2.13}$$

where $\gamma \approx 0.577215..$ is Euler's constant. We can also write this as that

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + z \cdot \sum_{n=1}^{\infty} \left(\frac{1}{n \cdot (z+n)} \right). \tag{2.14}$$

If we now apply (2.14) in the left hand side of (2.5) we get by some elementary simplifications that

$$\frac{\Gamma'(a)}{\Gamma(a)} - \frac{\Gamma'(a+b)}{\Gamma(a+b)} = \frac{b}{a \cdot (a+b)} - b \cdot \sum_{n=1}^{\infty} \frac{1}{(a+n) \cdot (a+b+n)}. \tag{2.15}$$

Hence we have proved

Proposition 2.2 *If* $\mathbf{p}$ *is* $B(a,b)$ *- distributed and if* a *and* b *are not negative integers, then*

$$E \log \mathbf{p} = \frac{b}{a \cdot (a+b)} - b \cdot \sum_{n=1}^{\infty} \frac{1}{(a+n) \cdot (a+b+n)}. \tag{2.16}$$

□

3 Variance of $\log \mathbf{p}$

Proposition 3.1 *If* $\mathbf{p}$ *is* $B(a,b)$ *- distributed and if* a *and* b *are not negative integers, then* $Var\,[\log \mathbf{p}]$ *can be expressed as a convergent sum.*

Proof: Let us consider the second moment of $\log \mathbf{p}$. We have

$$E \log^2 \mathbf{p} = \frac{1}{B(a,b)} \int_0^1 \log^2 p \cdot p^{a-1} \cdot (1-p)^{b-1} dp, \tag{3.1}$$

where $\log^2 p = (\log p)^2$. But since from (2.6)

$$\frac{d^2}{da^2} x^a = \log x \cdot \frac{d}{da} x^a = \log^2 x \cdot x^a \tag{3.2}$$

we get by the above (3.2) that

$$E \log^2 \mathbf{p} = \frac{1}{B(a,b)} \cdot \frac{\partial^2}{\partial a^2} B(a,b). \tag{3.3}$$

On the other hand using $\frac{\partial}{\partial a} B(a,b) = B(a,b) \cdot \frac{\partial}{\partial a} \log B(a,b)$ we get

$$\frac{\partial^2}{\partial a^2} B(a,b) = \frac{\partial}{\partial a} B(a,b) \cdot \frac{\partial}{\partial a} \log B(a,b) + B(a,b) \frac{\partial^2}{\partial a^2} \log B(a,b)$$

so that

$$\frac{1}{B(a,b)} \cdot \frac{\partial^2}{\partial a^2} B(a,b) = \frac{1}{B(a,b)} \frac{\partial}{\partial a} B(a,b) \cdot \frac{\partial}{\partial a} \log B(a,b) + \frac{\partial^2}{\partial a^2} \log B(a,b). \tag{3.4}$$

This yields

$$\frac{1}{B(a,b)} \cdot \frac{\partial^2}{\partial a^2} B(a,b) = \left(\frac{\partial}{\partial a} \log B(a,b) \right)^2 + \frac{\partial^2}{\partial a^2} \log B(a,b). \tag{3.5}$$

From (2.12) we get

$$\frac{\partial^2}{\partial a^2} \log B(a,b) = \frac{\partial}{\partial a} \frac{\Gamma'(a)}{\Gamma(a)} - \frac{\partial}{\partial a} \frac{\Gamma'(a+b)}{\Gamma(a+b)}. \tag{3.6}$$

This is a useful formula due to the fact that derivatives of the natural logarithm of the Gamma function (*polygamma functions*) have a known series representation. In fact it holds that if $L'(z) = \Gamma'(z)/\Gamma(z)$, then for $k \geq 2$

$$\frac{d^k}{dz^k} L(z) = (-1)^k (k-1)! \sum_{n=0}^{\infty} (z+n)^{-k}, \tag{3.7}$$

see [5, p. 467]. Thus we have

$$\frac{\partial}{\partial a} \frac{\Gamma'(a)}{\Gamma(a)} = \sum_{n=0}^{\infty} (a+n)^{-2} \tag{3.8}$$

and

$$\frac{\partial}{\partial a}\frac{\Gamma'(a+b)}{\Gamma(a+b)} = \sum_{n=0}^{\infty}(a+b+n)^{-2}. \tag{3.9}$$

Thus

$$\frac{\partial^2}{\partial a^2}\log B(a,b) = \sum_{n=0}^{\infty}\left((a+n)^{-2} - (a+b+n)^{-2}\right), \tag{3.10}$$

which equals

$$\frac{\partial^2}{\partial a^2}\log B(a,b) = \sum_{n=0}^{\infty}\frac{b^2+2ab+2bn}{((a+n)\cdot(a+b+n))^2}, \tag{3.11}$$

Hence from (3.3), (3.4) and (2.9) we get

$$\begin{aligned} E\log^2\mathbf{p} &= \frac{1}{B(a,b)}\cdot\frac{\partial^2}{\partial a^2}B(a,b) \\ &= \left(\frac{\partial}{\partial a}\log B(a,b)\right)^2 + \frac{\partial^2}{\partial a^2}\log B(a,b) \\ &= (E\log\mathbf{p})^2 + \sum_{n=0}^{\infty}\frac{b^2+2ab+2bn}{((a+n)\cdot(a+b+n))^2}. \end{aligned} \tag{3.12}$$

Next, we recall that

$$Var[\log\mathbf{p}] = \left(E\log^2\mathbf{p}\right) - (E\log\mathbf{p})^2.$$

Thus we have proved

$$Var[\log\mathbf{p}] = \sum_{n=0}^{\infty}\frac{b^2+2ab+2bn}{((a+n)\cdot(a+b+n))^2}. \tag{3.13}$$

□

4 Information Component

In view of the preceding (2.16 and 3.13) we can compute expectation and variance of $IC_{ij} = \log\frac{\mathbf{p}_{ij}}{\mathbf{p}_i\cdot\mathbf{p}_j}$ if $\mathbf{p}_{ij}, \mathbf{p}_i$ and $\mathbf{p}_j$ have their respective Beta distributions. However, when calculating the variance we can not expect $\mathbf{p}_{ij}$, $\mathbf{p}_i$ and $\mathbf{p}_j$ to be independent which is why we need the joint distributions to compute the covariances $cov(\log\mathbf{p}_{ij}, \log\mathbf{p}_i)$, $cov(\log\mathbf{p}_{ij}, \log\mathbf{p}_j)$ and $cov(\log\mathbf{p}_i, \log\mathbf{p}_j)$. Expectation and variance of IC_{ij} can then be expressed as

$$E[IC_{ij}] = E\log\mathbf{p}_{ij} - E\log\mathbf{p}_i - E\log\mathbf{p}_j, \tag{4.14}$$

$$\begin{aligned} Var[IC_{ij}] &= Var\log\mathbf{p}_{ij} + Var\log\mathbf{p}_i + Var\log\mathbf{p}_j + \\ &\quad -2cov(\log\mathbf{p}_{ij}, \log\mathbf{p}_i) - 2cov(\log\mathbf{p}_{ij}, \log\mathbf{p}_j) + 2cov(\log\mathbf{p}_i, \log\mathbf{p}_j). \end{aligned} \tag{4.15}$$

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